

On a Family of Norm Form Cubic Surfaces

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Abstract

In a 1975 paper of Birch and Swinnerton-Dyer, a number of explicit norm form cubic surfaces are shown to fail the Hasse Principle. They make a correspondence between this failure and the Brauer–Manin obstruction, recently discovered by Manin.

Using techniques developed in the ensuing 40 years, we show that a much wider class of norm form cubic surfaces have a Brauer–Manin obstruction to the Hasse principle, thus verifying the Colliot-Thélène–Sansuc conjecture for infinitely many cubic surfaces.

1 Introduction

Suppose X is a smooth projective variety over a number field k . Denote by $X(k)$ and $X(\mathbf{A}_k)$ the rational and adélic points of X , respectively. The natural inclusion of k into \mathbf{A}_k gives $X(k) \neq \emptyset \Rightarrow X(\mathbf{A}_k) \neq \emptyset$. The variety X satisfies the *Hasse principle* if the converse is true, $X(\mathbf{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$. There are many examples of varieties which do not satisfy the Hasse Principle; Cassels and Guy, in [2], provided one of the original counterexamples,

$$5x^3 + 12y^3 + 9z^3 + 10w^3 = 0. \quad (1)$$

As a cohomological generalization of quadratic reciprocity, Manin constructs, in [9] the *Brauer Set* $X(\mathbf{A}_k)^{\text{Br}}$ which lies between $X(k)$ and $X(\mathbf{A}_k)$. We say that X has a *Brauer–Manin obstruction to the Hasse Principle* if $X(\mathbf{A}_k)^{\text{Br}} = \emptyset$ while $X(\mathbf{A}_k) \neq \emptyset$. Around this time, Birch and Swinnerton-Dyer, in [1], considered counterexamples to the Hasse Principle for rational surfaces via very direct arguments. They comment that Manin’s method should apply and provide a brief sketch to this effect. We will examine the cubic surfaces constructed by Birch and Swinnerton-Dyer:

Let K_0/k be a non-abelian cubic extension, and L/k its algebraic closure. Suppose K/k is the unique quadratic extension which lies in L . We will assume that $(1, \phi, \psi)$ are any linearly independent generators for K_0/k , and K/k is generated by θ . Then consider the diophantine equation given by

$$m \prod_{i=0}^2 (ax + by + \phi_i z + \psi_i w) = (cx + dy)(x + \theta y)(x + \bar{\theta} y), \quad (2)$$

where the ϕ_i, ψ_i are the Galois conjugates of ϕ_0, ψ_0 and $\bar{\theta}$ is that of θ over k , and m, a, b, c, d are suitably chosen k -rational integers.

Birch and Swinnerton-Dyer show that as long as a, b, c, d have “certain” divisibility properties, these surfaces do not satisfy the Hasse Principle. This is done by considering a rational solution $[x, y, z, w]$ and examining the possible factorizations of the ideal $(x + \theta y)$ in \mathcal{O}_K . They find two possible reasons the Hasse Principle may fail and give an example computation of the Brauer–Manin obstruction for each. This paper re-examines the BSD surfaces with the machinery and language of present-day Geometry and Class Field Theory.

The following theorem follows from our main theorem.

Theorem 1.1. *Take $k = \mathbb{Q}$, L/K unramified, and the ϕ_i and ψ_i to be integral units with the minimal polynomial of ψ_i/ϕ_i being separable modulo 3. Suppose p is a prime for which $p\mathcal{O}_L = \mathcal{P}_1\mathcal{P}_2$ such that $p \nmid \theta\bar{\theta}$. Then the variety defined by*

$$\prod_{i=0}^2 (x + \phi_i z + \psi_i w) = py(x + \theta y)(x + \bar{\theta} y)$$

has a Brauer–Manin obstruction to the Hasse Principle.

1.1 The Colliot-Thélène–Sansuc Conjecture

After the paper of Birch and Swinnerton-Dyer, Colliot-Thélène and Sansuc conjectured that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for arbitrary smooth projective geometrically rational surfaces [5, Questions j₁, k₁, page 233]. Some motivation for this conjecture came from the study of conic bundles.

In 1987, Colliot-Thélène, Kanevsky and Sansuc [4] systematically studied diagonal cubic surfaces over \mathbb{Q} having integral coefficients up to 100, verifying the conjecture for each one of these surfaces. They were the first to prove that the Cassels and Guy cubic (1) had a Brauer–Manin obstruction.

This conjecture has been extended by Colliot-Thélène to all rationally connected varieties [3]; evidence of this generalization has been seen recently in works of Harpaz and Wittenberg (see e.g. [7]).

1.2 Outline

The notation for the paper will be fixed in section 2. We will explicitly describe the Brauer group for the BSD cubic surfaces in section 3. This computation will exploit the exceptional geometry of cubic surfaces and the previous results of Corn and Swinnerton-Dyer.

In section 4, there is a lemma arguing the existence of an adélic point for a family of surfaces followed by general computations of the Brauer set. Theorems 4.2 and 4.3 show that we only need to consider certain primes which divide the coefficient d .

Lastly, in section 5, we first look back at an example given in [1] and verify that its obstruction is given by the results of section 4. A second example with a Brauer–Manin obstruction given by two non-zero invariant summands is then presented.

2 Setup and notation

Let k be a number field with absolute Galois group G_k . Take L/k any Galois extension with $\text{Gal}(L/k) \simeq S_3$. Fix K/k as the unique quadratic extension of k in L . Let \mathcal{O}_F be the ring of integers for the field F .

Lemma 2.1. *Every BSD cubic (2) is isomorphic to one of the form*

$$\prod_{i=0}^2 (x + \phi_i z + \psi_i w) = dy(x + \theta y)(x + \bar{\theta} y), \quad (3)$$

where $d \in \mathcal{O}_k$, and $\{\phi_0, \phi_1, \phi_2\}, \{\psi_0, \psi_1, \psi_2\} \subseteq \mathcal{O}_L$ and $\{\theta, \bar{\theta}\} \subseteq \mathcal{O}_K$ are respective Galois conjugates over k with $(1, \phi_i, \psi_i)$ being a k -basis for a degree 3 extension of k .

Proof. There is an isomorphism of varieties given by

$$[x : y : z : w] \mapsto \left[ax + by : \frac{cx + dy}{(d - c\theta)(d - c\bar{\theta})} : z : w \right],$$

from the surface (2), to the surface

$$m(ad - bc)^2 \prod_{i=0}^2 (x + \phi_i z + \psi_i w) = ((d - c\theta)(d - c\bar{\theta}))^2 y(x + \theta' y)(x + \bar{\theta}' y),$$

where $\theta' = (-b + a\theta)(d - c\bar{\theta})$. A subsequent isomorphism given by scaling variables results in (3). \square

Let X to be a variety of the form (3) and let \bar{X} be the base change of X to the separable closure of k . Take $\text{Pic } \bar{X}$ to be the Picard group of \bar{X} . For a fixed $\mathcal{A} \in \text{Br } X := H_{\text{ét}}^2(X, \mathbb{G}_m)$, there is a commutative diagram

$$\begin{array}{ccccccc} X(k) & \hookrightarrow & X(\mathbf{A}_k) & & & & \\ \downarrow \text{ev}_{\mathcal{A}} & & \downarrow \text{ev}_{\mathcal{A}} & \searrow \Phi_{\mathcal{A}} & & & \\ 0 & \longrightarrow & \text{Br } k & \longrightarrow & \oplus_v \text{Br } k_v & \xrightarrow{\text{inv}} & \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \end{array}$$

where $\text{ev}_{\mathcal{A}}$ is the specialization of \mathcal{A} and $\text{inv} = \sum_v \text{inv}_v$ is the *invariant map*. The *Brauer Set* is defined as $X(\mathbf{A}_k)^{\text{Br}} := \bigcap_{\mathcal{A} \in \text{Br } X} \Phi_{\mathcal{A}}^{-1}(0)$.

There is an inclusion of $\text{Br } X$ into $\text{Br } k(X)$, so elements of $\text{Br } X$ can be realized as Azumaya Algebras over the field $k(X)$. Assume F_1/F_2 are fields, $\text{Gal}(F_1/F_2) = \langle \sigma \rangle$ is cyclic of order n and $a \in F_2^*$, then the *cyclic algebra* $(F_1/F_2, a) \in \text{Br } F_2$ is defined to be the quotient $F_1[T]_{\sigma}/(T^n - a)$. Here $F_1[T]_{\sigma}$ is the twisted polynomial ring, i.e. $Tb = \sigma(b)T$ for all $b \in F_1$.

3 Computing the Brauer group

Since X is rational, $\ker(\mathrm{Br} X \rightarrow \mathrm{Br} \overline{X}) = \mathrm{Br} X$, [10, Thm. 42.8], and there is an isomorphism,

$$\mathrm{Br} X / \mathrm{Br} k \xrightarrow{\sim} H^1(G_k, \mathrm{Pic} \overline{X}). \quad (4)$$

Moreover, $\Phi_{\mathcal{A}}$ factors through this quotient. Therefore it will be sufficient to calculate this finite group rather than determining the entire group $\mathrm{Br} X$.

Theorem 3.1. *Either $H^1(G_k, \mathrm{Pic} \overline{X}) \simeq \mathbb{Z}/3\mathbb{Z}$ or $H^1(G_k, \mathrm{Pic} \overline{X})$ is trivial.*

Before proving this result we must make the following definition.

Definition. A *nine* on X is a set consisting the three skew curves together with six curves intersecting exactly two of those three. A *triple-nine* is a partitioning of the 27 exceptional curves on X into three nines.

Proof. It has been shown in [12, Lem. 6] and [6, Lem. 1.3.22] that non-trivial $\mathcal{A} \in H^1(G_k, \mathrm{Pic} \overline{X})[3]$ correspond to triple-nines on \overline{X} such that every nine is G_k stable but no skew triple is itself G_k stable.

There are 9 lines, $L_{i,j}$, defined by $0 = x + \phi_i z + \psi_i w$ and

$$0 = \begin{cases} y & \text{if } j = 0, \\ x + \theta y & \text{if } j = 1, \\ x + \overline{\theta} y & \text{if } j = 2, \end{cases}$$

and 18 lines, $L_{(i,j,k),n}$ given by $z = Ax + By$ and $w = Cx + Dy$ such that A, B, C , and D satisfy the system of equations

$$\begin{cases} 1 + A\phi_i + C\psi_i = 0, \\ \theta(1 + A\phi_j + C\psi_j) = (B\phi_j + D\psi_j), \\ \overline{\theta}(1 + A\phi_k + C\psi_k) = (B\phi_k + D\psi_k), \\ (B\phi_0 + D\psi_0)(B\phi_1 + D\psi_1)(B\phi_2 + D\psi_2) = d\theta\overline{\theta}. \end{cases}$$

Let L'/k be the field of definition for the 27 lines. A triple-nine for which the individual nines are fixed by G_k is

$$\begin{pmatrix} L_{0,0} & L_{1,1} & L_{2,2} \\ L_{1,2} & L_{2,0} & L_{0,1} \\ L_{2,1} & L_{0,2} & L_{1,0} \end{pmatrix}, \begin{pmatrix} L_{(0,1,2),0} & L_{(0,1,2),1} & L_{(0,1,2),2} \\ L_{(1,2,0),0} & L_{(1,2,0),1} & L_{(1,2,0),2} \\ L_{(2,0,1),0} & L_{(2,0,1),1} & L_{(2,0,1),2} \end{pmatrix}, \begin{pmatrix} L_{(0,2,1),0} & L_{(0,2,1),1} & L_{(0,2,1),2} \\ L_{(1,0,2),0} & L_{(1,0,2),1} & L_{(1,0,2),2} \\ L_{(2,1,0),0} & L_{(2,1,0),1} & L_{(2,1,0),2} \end{pmatrix}. \quad (5)$$

The Galois group G_k permutes the first nine, fixing no skew triple. The rows of the second two nines will be permuted via the permutation action on the roots (ϕ_0, ϕ_1, ϕ_2) . The action of G_k on the columns of the second two nines will determine whether or not any skew triple is fixed. If $[L' : L] = 1$ or 2 , then $H^1(G_k, \mathrm{Pic} \overline{X})$ is trivial as some skew triples of the later 2 nines will be fixed by G_k . Otherwise there is a non-trivial $\mathcal{A} \in H^1(G_k, \mathrm{Pic} \overline{X})[3]$. The first nine in the list above must appear in every triple nine with the specified G_k action, so all possible triple nines have been found. \square

The map in (4) is generically difficult to invert. We achieve this via the following result of Swinnerton-Dyer and Corn.

Lemma 3.2 ([13, Lem. 2], [6, Prop. 2.2.5]). *Non-trivial elements of $H^1(G_k, \mathrm{Pic} \overline{X})[3]$ correspond to $\mathcal{A} \in \mathrm{Br} X$ such that $\mathcal{A} \otimes_k K = (L(X)/K(X), f)$ in the group $\mathrm{Br} X_K / \mathrm{Br} K$, where $\mathrm{div}(f) = N_{L/K}(D)$ in $\mathrm{Div} X_L$ for some non-principle divisor D .*

Corollary 3.3. *If $H^1(G_k, \mathrm{Pic} \overline{X}) \simeq \mathbb{Z}/3\mathbb{Z}$ then it is generated by an algebra \mathcal{A} such that $\mathcal{A} \otimes_k K \simeq (L(X)/K(X), \frac{x+\theta y}{y})$.*

Proof. Take $D = L_{0,0} + L_{1,1} + L_{1,0} - \mathrm{div}(y)$. Then

$$\begin{aligned} \mathrm{Norm}_{L/K}(D) &= (L_{0,0} + L_{1,1} + L_{1,0}) + (L_{1,0} + L_{2,1} + L_{2,0}) + (L_{2,0} + L_{0,1} + L_{0,0}) - 3\mathrm{div}(y), \\ &= L_{1,1} + L_{2,1} + L_{0,1} - \mathrm{div}(y), \\ &= \mathrm{div}(x + \theta y) - \mathrm{div}(y), \\ &= \mathrm{div}\left(\frac{x + \theta y}{y}\right). \end{aligned} \quad \square$$

4 Invariant Map Computations

Since the $\mathcal{A} \in \text{Br } X / \text{Br } k$ are explicit, one may compute the map $\phi_{\mathcal{A}}$ more easily. However, before doing so, we would like to verify the existence of an adélic point.

Lemma 4.1. *In addition to the setup of section 2, assume the following are true:*

1. L/K is unramified,
2. $\phi_0\phi_1\phi_2 = \psi_0\psi_1\psi_2 = \pm 1$,
3. the minimal polynomial for ψ_i/ϕ_i over k is separable modulo $\mathfrak{p} \mid 3\mathcal{O}_k$, and
4. if $\mathfrak{p} \mid d\mathcal{O}_L$ with $\mathfrak{p}\mathcal{O}_L = \mathcal{P}_1\mathcal{P}_2$ then $v_1(d) \leq v_1(\theta)$ with $v_1(\bar{\theta}) = 0$, equivalently $v_2(d) \leq v_2(\bar{\theta})$ with $v_2(\theta) = 0$, where v_i is the valuation corresponding to \mathcal{P}_i .

Then $X(\mathbf{A}_k) \neq \emptyset$.

Proof. In most cases, the scheme given by $X \cap V(x)$ will be a genus 1 curve and will subsequently have a $k_{\mathfrak{p}}$ point by the Hasse bound. This will be the case whenever $\mathfrak{p} \nmid 3d\mathcal{O}_k$.

Suppose $\mathfrak{p} \mid 3\mathcal{O}_k$ and $\mathfrak{p} \nmid d\mathcal{O}_k$. Then $X \cap V(x) \rightarrow \mathbb{P}^1$ defined by $[0 : y : z : w] \mapsto [z : w]$ is one-to-one and surjective on $k_{\mathfrak{p}}$ points. Assumption 3 provides that at least one of these points is smooth.

For the primes \mathfrak{p} of k dividing d , to show $X(k_{\mathfrak{p}}) \neq \emptyset$, it will suffice to find a $K_{\mathcal{P}}$ point for each prime \mathcal{P} of K dividing \mathfrak{p} . This is a result of the fact that on cubic surfaces the existence of $k_{\mathfrak{p}}$ -rational points is equivalent to that for quadratic extensions of $k_{\mathfrak{p}}$ (cf. [6, Lem. 1.3.25]).

If $\mathcal{P} \mid d\mathcal{O}_K$ and \mathcal{P} splits over L then $X_{\mathcal{P}}$ is the union of 3 lines all defined over $K_{\mathcal{P}}/\mathcal{P}$ and has many $K_{\mathcal{P}}$ points.

Lastly, suppose $\mathcal{P} \mid d\mathcal{O}_K$ and \mathcal{P} remains prime in L . Then we are in the case of $v_{\mathcal{P}}(d) \leq v_{\mathcal{P}}(\theta)$. Then consider X' given by

$$d \prod_{i=0}^2 (x + \phi_i z + \psi_i w) - y \left(x + \frac{\theta}{d} y \right) (dx + \bar{\theta} y),$$

which is isomorphic to X . Note that this equation has $\mathcal{O}_{\mathcal{P}_1}$ coefficients since $v_1(\theta) \geq v_1(d)$. Modulo \mathcal{P}_1 , the defining equation for X' becomes

$$X'_1 : \bar{\theta}_1 y^2 \left(x + \left(\frac{\theta}{d} \right)_1 y \right),$$

where $\bar{\theta}_1$ and $(\frac{\theta}{d})_1$ are the restriction of the respective constants to the quotient $\mathcal{O}_{\mathcal{P}_1}/\mathcal{P}_1$. The surface X'_1 has a smooth point $[\theta_1/d : -1 : 1 : 1]$ which will lift to a $K_{\mathcal{P}_1}$ point, $[x_0 : y_0 : z_0 : w_0] \in X'(K_{\mathcal{P}_1})$. Via the isomorphism, we have $[dx_0 : y_0 : dz_0 : dw_0] \in X(K_{\mathcal{P}_1})$. \square

Remark. In the case of $v_1(d) > v_1(\theta)$, a similar argument can be made with the additional assumption of the surjectivity of the cube map in $\mathcal{O}_{\mathcal{P}_1}/\mathcal{P}_1$.

Of course Lemma 4.1 is not comprehensive; there are surfaces in the class which have adélic points but do not satisfy the conditions listed above. The intention of this lemma is to provide proof that there are indeed infinitely many surfaces of this form which have an adélic point.

Remark. If $\text{Br } X / \text{Br } k$ is trivial, then the triple nine as in (5) will have enough fixed skew triples to build a set of six skew lines which is G_K -stable. We blow down X_K along these six skew lines to obtain a degree 9 del Pezzo surface X' defined over K . It is well-known that degree 9 del Pezzo surfaces satisfy the Hasse principle. So by the Lang-Nishimura lemma X_K must also satisfy the Hasse principle, and as mentioned before, X satisfies the Hasse principle if and only if X_K satisfies the Hasse principle (cf. [6, Lem. 1.3.25]). Therefore, we will only consider surfaces X for which $\text{Br } X / \text{Br } k \simeq \mathbb{Z}/3\mathbb{Z}$.

There is a classical formula for inv_v provided L_v/K_v is unramified given by the local Artin map. That is, for all places v unramified in L/K ,

$$\text{inv}_v((L_v/K_v, f(P_v))_{\sigma}) = \frac{ij}{k} \pmod{1},$$

where $i = v_v(f(P))$, $\sigma^j = \text{Frob}_{L_v/K_v}$, and $k = [L_v : K_v]$ (cf. [11, XIV.2]).

Theorem 4.2. *Assume the notation of section 2. Suppose v is a finite place of K which is unramified in L/K such that $v_v(d) = 0 \pmod{3}$, and that θ or $\bar{\theta}$ has valuation 0. Then $\text{inv}_v(\mathcal{A}_K(P_v)) = 0$ for all $P \in X(\mathbf{A}_K)$. Moreover, $\text{inv}_{\infty}(\mathcal{A}_K(P_{\infty})) = 0$.*

Proof. (The structure of this proof follows that of [8, III.5.18].) In the infinite case, we must have $\text{inv}_\infty(\mathcal{A}_K(P)) = 0$, as $[L : K] = 3$ and $\text{inv}_\infty(\mathcal{A}_K(P_\infty)) = 0$ or $1/2$.

Suppose that v splits completely in L . Then $L_v = K_v$ and $(L_v/K_v, f(P_v))$ is trivial, so $\text{inv}_v(\mathcal{A}_K(P_v)) = 0$.

If v remains prime in L then $[L_v : K_v] = 3$. Take $P_v = [x_0 : y_0 : z_0 : w_0] \in X(K_v)$. Via scaling, assume that x_0, y_0, z_0 and w_0 are integral and at least one has valuation 0. Since $\prod_{i=0}^2 (x_0 + \phi_i z_0 + \psi_i w_0)$ is a norm from L to K , $y_0 = 0$ would imply $x_0 = z_0 = w_0 = 0$, which is not possible. Thus $y_0 \neq 0$. In particular, $f = \frac{x+\theta y}{y}$ is defined for all $P_v \in X(K_v)$.

For simplicity, set $v = v_v$ and $N = \prod_{i=0}^2 (x + \phi_i z + \psi_i w)$. If $v(N) = 0$, then $v(y) = v(x + \theta y) = 0$. Hence $\text{inv}(\mathcal{A}_K(P_v)) = 0$. On the other hand, suppose $v(N) > 0$. Since N is a norm on the residue class fields, $v(x_0), v(z_0), v(w_0) > 0$. Hence $v(y_0) = 0$. In fact, $3 \mid v(N)$. Thus, $v(d) + v(x + \theta y) + v(x + \bar{\theta} y) \equiv 0 \pmod{3}$. However, $v(x + \theta y) = 0$ or $v(x + \bar{\theta} y) = 0$, since v does not divide both θ and $\bar{\theta}$. In particular, $v(x + \theta y) \equiv 0 \pmod{3}$ and

$$\text{inv}_v(\mathcal{A}_K(P_v)) = 0. \quad \square$$

Remark. This result should be expected, because unramified primes of good reduction produce a trivial invariant computation.

Given the result of Theorem 4.2, in all cases where L/K is unramified, we simply need to consider the places of k over which d has valuation that is non-zero modulo 3. The following theorem provides a sample of the types of Brauer–Manin obstructions we may now construct for the surfaces X .

Theorem 4.3. *With the notation of section 2. Suppose L/K is unramified. Fix θ so that no primes of \mathcal{O}_K divide both θ and $\bar{\theta}$. Let \mathfrak{p} be a prime of \mathcal{O}_k such that $\mathfrak{p}^n \mid (d)$ for some $n \not\equiv 0 \pmod{3}$ which also divides $\theta\bar{\theta}$. Suppose all other primes dividing (d) split in L/K . If $X(\mathbf{A}_K) \neq \emptyset$ and $\mathfrak{p} = \mathcal{P}_1\mathcal{P}_2$ in \mathcal{O}_K and in \mathcal{O}_L , then $\sum_v \text{inv}_v(\mathcal{A}_K(P)) \neq 0$.*

Proof. From the statement and proof of Theorem 4.2, we need only consider the primes \mathcal{P}_1 and \mathcal{P}_2 of \mathcal{O}_K that lie above \mathfrak{p} . Via our assumption that no primes divide both θ and $\bar{\theta}$, we can assume that $\mathcal{P}_1 \mid \theta$ and $\mathcal{P}_2 \nmid \theta$. Take $v_i = v_{\mathcal{P}_i}$ to be the respective valuation maps. As $v_i(dy(x + \theta y)(x + \bar{\theta} y)) > 0$, we must be in the case that $v_i(x_0), v_i(y_0), v_i(z_0) > 0$ and $v_i(y_0) = 0$. Then $v_2(x + \theta y) = 0$, so $\text{inv}_{\mathcal{P}_2}(\mathcal{A}(P)) = 0$. On the other hand $v_1(x + \theta y) = v_1(dy(x + \theta y)(x + \bar{\theta} y)) - v_1(d) \equiv -v_1(d) \pmod{3}$. In particular $\text{inv}_{\mathcal{P}_1}(\mathcal{A}(P)) = 1/3$ or $2/3$. Thus

$$\sum \text{inv}_v(\mathcal{A}(P)) = \text{inv}_{\mathcal{P}_1}(\mathcal{A}(P)) \neq 0. \quad \square$$

This theorem provides a jumping off point for similar results. One may consider the case where more places divide d , and examples of most forms can be computed immediately.

5 Examples

Examples that fit the situation of this Theorem 4.3 are easy to come by. Given any L/K unramified we can find many such θ . Then it is a quick check via Hensel's Lemma and the Weil Conjectures to show that there is an adelic point. In fact the original example of BSD fits this case.

Example. Suppose $\theta' = \frac{1}{2}(1 + \sqrt{-23})$ and ϕ_i so that $\phi_i^3 = \phi_i + 1$ and $\psi_i = \phi_i^2$. Define X_{BSD} by

$$2 \prod_{i=0}^2 (x + \phi_i z + \phi_i^2 w) = (x - y)(x + \theta' y)(x + \bar{\theta}' y).$$

Via the isomorphisms above, we have the isomorphic X given by

$$\prod_{i=0}^2 (x + \phi_i z + \psi_i^2 w) = 32y(x + \theta y)(x + \bar{\theta} y),$$

where $\theta = -\theta' - 6$.

We find that X has adelic points but no rational points. Moreover, X has a Brauer–Manin obstruction to rational points as described in Theorem 4.3.

There are few published examples where the invariant map has two or more non-zero summands. Given the theorems above, examples of this can be found quickly.

Example. Suppose the ϕ_i satisfy $\phi_i^3 + \phi_i + 1 = 0$ and $\theta, \bar{\theta}$ are the roots of $T^2 - 4T + 35$.
Then

$$X : \prod_{i=0}^2 (x + \phi_i z + \psi_i w) = 5^2 \cdot 7y(x + \theta y)(x + \bar{\theta} y),$$

has a Brauer–Manin obstruction to the Hasse Principle with the invariant map being

$$1/3 + 1/3 \quad \text{or} \quad 2/3 + 2/3,$$

depending on the choice of algebra \mathcal{A} .

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